# On the equilibrium state of random walkers in random environments: analytical results



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# On the equilibrium state of random walkers in random environments: analytical results

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**Abstract.** We study equilibrium properties of random walkers in one-dimensional random environments of finite length $L$. From an exact expression for the quenched average of the free energy we derive analytical results for all cumulants and all Rényi entropies of the equilibrium distribution. In contrast to the finite variance of a typical non-equilibrium distribution in the unbiased situation, we find that in equilibrium the disorder averaged variance diverges with the size of the system as $L^{3/2}$.

## 1. Introduction

Random walks in random environments (RWRE) are simplified models for the motion of particles or states in random media. Their relevance for various physical systems is reviewed, for example, in [1–3]. After initial work by mathematicians [4, 5], especially that of Sinai [6] and Golosov [7] on the anomalous time dependence of the mean-value and the mean-square displacement in the unbiased, one-dimensional case, many other relevant quantities were investigated for this model in the physical literature. Examples are the mean velocity and the diffusion constant in the presence of a global bias [8], $1/f$-noise in such systems [9], passage time distributions [10] and its multifractal properties [11, 12], and recently the response to concentration gradients [13, 14]. There is continuing interest in these systems on the physical side because of questions of self-averaging [15] and aging [16, 17]. On the mathematical side, the fundamental question of recurrence of such random walks, even in one dimension, has only been partially solved [18, 19]. For the status in higher dimensions we refer to the above-mentioned review articles.

We became interested in these systems after realizing that many dissipative chaotic systems exhibit a dynamical localization phenomenon [20], which should be regarded as a generalization of the Golosov phenomenon proved in [7] (see also [2]). Subsequently, the question arose whether such RWREs can also be realized in Hamiltonian systems. This was answered affirmatively in [21] by constructing inhomogeneous chains of area-preserving Baker maps showing the same phenomenon. This construction utilizes the equilibrium distribution for a RWRE of finite length. An important question in this context is whether the mean-square displacement of a RWRE in equilibrium remains bounded or not if the system length $L$ is increased. The Golosov phenomenon refers to the non-equilibrium situation where one considers the evolution of a distribution in the infinite system. It has been shown that for a typical disorder realization $\sigma^2(t, L = \infty)$ remains bounded [7], but

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is this true also for the equilibrium value $\sigma^2(t=\infty, L)$? It turns out that there exists little work on the equilibrium distribution, which in one dimension is simply the Boltzmann–Gibbs distribution. We are aware only of an investigation by Grassberger and Leuverink [22] of the Rényi entropies [23] criticizing previous work [24] and a discussion by Parisi [25] in the context of directed polymers (see also [16]).

## 2. Model considerations and analytical results

In the discrete formulation a RWRE is governed by the evolution equation

$$\pi_j(t+1) = \sum_i \pi_i(t) p_{ij} \tag{2.1}$$

where $\pi_i(t)$ is the probability to be on site $i$ at discrete time $t$ and $p_{ij}$ are transition probabilities, which are time-independent random variables. The latter provide the quenched random environment. In the following we consider only systems with nearest-neighbour transitions $p_{ij} = p_i \delta_{j,i+1} + (1-p_i)\delta_{j,i-1}$, $1 < i < L$, and with reflecting boundary conditions at $i = 1$ and $i = L$, i.e. $p_{12} = p_{L,L-1} = 1$. The stationary distribution $\rho_i$ is obtained from detailed balance $\rho_i p_i = \rho_{i+1}(1-p_{i+1})$ as

$$\rho_i = \rho_1 \prod_{l=1}^{i-1} \frac{p_l}{1-p_{l+1}} \tag{2.2}$$

where $\rho_1$ is determined by the normalization

$$(\rho_1)^{-1} = Z_L = 1 + \sum_{i=2}^{L} \prod_{l=1}^{i-1} \frac{p_l}{1-p_{l+1}}. \tag{2.3}$$

The transition probabilities $p_l$ are chosen as independent identically distributed random variables obeying

$$\overline{\ln^2(p_l/(1-p_l))} - \overline{\ln(p_l/(1-p_l))}^2 = 2\alpha \qquad \text{and} \qquad \overline{\ln(p_l/(1-p_l))} = \mu\alpha.$$

The overbar denotes a quenched average, i.e. the average over realizations of the random environment. Rewriting (2.2) as $\rho_i = \exp(-\beta V_i)/Z_L$, one sees that $\beta V_i$ is essentially the trace of a discrete random walk with independent increments $\ln(p_l/(1-p_l))$, which is characterized by a diffusion constant $\alpha$ and a bias $\mu$. We are mainly interested in the case $\mu = 0$, which is usually referred to as the 'Sinai case'. In our numerical simulations presented in [26], we calculated quantities like the variance of the equilibrium distribution $K_L^{(2)} = \sum_{i=1}^{L} i^2\rho_i - (\sum_{i=1}^{L} i\rho_i)^2$, quenched averages $\overline{K_L^{(2)}}$, or averages of the form $\overline{\ln K_L^{(2)}}$. The random variables $p_l$ in these simulations were, for simplicity, always drawn from a binary, unbiased distribution, i.e. $p_l$ takes the values $p$ and $1-p$ with equal probability. Even for this simple, discrete system it seems to be impossible to calculate any of the above quantities exactly. We, therefore, consider in our analytic treatment the continuum version of the above model.

In the continuum limit the discrete potential $V_i$ is replaced by a continuous potential $V(x)$, which varies like a Brownian path, characterized by the same diffusion constant and bias as the discrete potential. This replacement affects only small scales, where the discreteness becomes visible. We are, however, interested in large-scale properties, i.e. properties which become macroscopic in the limit $L \to \infty$. We will thus observe only coarse-grained properties where the discreteness is irrelevant. In the continuum limit the partition function $Z_L$ of (2.3) is given by†

$$Z_L = \int_0^L \mathrm{d}x \exp\left( - \sqrt{2\alpha} \int_0^x \xi(y)\, \mathrm{d}y - \mu\alpha x \right) \tag{2.4}$$

where $\xi(y)$ is an uncorrelated, Gaussian random variable with zero mean and unit variance $\overline{\xi(y)\xi(y')} = \delta(y - y')$. From expression (2.4) we may obtain the cumulants of the equilibrium distribution $\rho(x) = \exp[-\beta V(x)]/Z_L$ from the derivatives with respect to $\mu$ of the cumulant generating function $\ln Z_L$ and accordingly their quenched averages as

$$\overline{K_L^{(m)}} = (-\alpha)^{-m} \frac{\partial^m}{\partial \mu^m} \overline{\ln Z_L}. \tag{2.5}$$

Thus one has to calculate the quenched average of the free energy of the system of length $L$. This is a non-trivial task. We can, however, rely on recent results for the generating function $\Phi_L(p) = \overline{\exp(-pZ_L)}$ by Monthus and Comtet [14]. They obtained an expression consisting of several terms

$$\Phi_L(p) = \Phi_L^{(\mathrm{c})}(p) + \sum_{0 \leqslant k \leqslant \mu/2} \Phi_L^{(k)}(p) \tag{2.6}$$

with

$$\Phi_L^{(\mathrm{c})}(p) = \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-(\alpha L/4)(\mu^2+s^2)} s \sinh(\pi s) \left| \Gamma\left( -\frac{\mu}{2} + \mathrm{i}\frac{s}{2} \right) \right|^2 \left( \frac{p}{\alpha} \right)^{\mu/2} K_{is}\left( 2\sqrt{\frac{p}{\alpha}} \right) \tag{2.7}$$

and

$$\Phi_L^{(k)}(p) = \mathrm{e}^{-\alpha L k(\mu-k)} \frac{2(\mu - 2k)}{k!\Gamma(1+\mu-k)} \left( \frac{p}{\alpha} \right)^{\mu/2} K_{\mu-2k}\left( 2\sqrt{\frac{p}{\alpha}} \right). \tag{2.8}$$

$\Gamma(x)$ and $K_\mu(x)$ denote the gamma function and modified Bessel functions of order $\mu$, respectively [27]. The parts $\Phi_L^{(k)}(p)$ are present only for $\mu \geqslant 0$ and stem from the discrete part of the spectrum of an associated Fokker–Planck operator, while $\Phi_L^{(\mathrm{c})}(p)$, due to a continuous branch of the spectrum, is always present [14]. From these expressions for $\Phi_L(p)$ one obtains $\overline{\ln Z_L}$ by the well known representation

$$\overline{\ln Z_L} = \int_0^\infty \mathrm{d}p \frac{1}{p} [\mathrm{e}^{-p} - \Phi_L(p)]. \tag{2.9}$$

Let us first calculate this average for $\mu > 0$. According to (2.6) and (2.9) we also have for $\overline{\ln Z_L}$ several contributions

$$\overline{\ln Z_L} = \overline{\ln Z_L^{(\mathrm{c})}} + \sum_{0 \leqslant k \leqslant \mu/2} \overline{\ln Z_L^{(k)}}. \tag{2.10}$$

The first part is simply given by

$$\overline{\ln Z_L^{(\mathrm{c})}} = -\int_0^\infty \mathrm{d}p \frac{1}{p} \Phi_L^{(\mathrm{c})}(p).$$

For $\mu > 0$ the $p$- and the $s$-integration of (2.7) can be interchanged and one obtains

$$\overline{\ln Z_L^{(\mathrm{c})}} = -2 \int_0^\infty \mathrm{d}s \frac{1}{\mu^2 + s^2}\, \mathrm{e}^{-(\alpha L/4)(\mu^2+s^2)} \frac{s \sinh(\pi s)}{\cosh(\pi s) - \cos(\pi \mu)}. \tag{2.11}$$

† The parametrization is chosen in accordance with [14].

The second contribution is the part

$$\overline{\ln Z_L^{(0)}} = \int_0^\infty \mathrm{d}p \frac{1}{p}[\mathrm{e}^{-p} - \Phi_L^{(0)}(p)]$$

which can be rewritten as

$$\lim_{x\to 0} 2\int_x^\infty \mathrm{d}y \frac{1}{y}\left[\exp(-\alpha y^2) - \frac{2}{\Gamma(\mu)} y^\mu K_\mu(2y)\right].$$

This integral can be evaluated [28] to yield simply

$$\overline{\ln Z_L^{(0)}} = -\ln\alpha - \Psi(\mu) \tag{2.12}$$

where $\Psi(\mu) = \partial \ln\Gamma(\mu)/\partial\mu$ is the Digamma function. For $\mu > 2k$, $k = 1, 2, \ldots,$ one obtains additional contributions

$$\overline{\ln Z_L^{(k)}} = -\int_0^\infty \mathrm{d}p \frac{1}{p}\Phi_L^{(k)}(p)$$

which evaluate to

$$\overline{\ln Z_L^{(k)}} = -\mathrm{e}^{-\alpha L k(\mu-k)} \frac{\mu - 2k}{k(\mu - k)}. \tag{2.13}$$

In the following we will show that in the limit $\mu \to 0$ each of the terms $\overline{\ln Z_L^{(0)}}$ and $\overline{\ln Z_L^{(c)}}$ becomes divergent, but in such a way that in their sum the singularities cancel exactly. This means that the results (2.10)–(2.13) can be analytically continued from $\mu > 0$ to $\mu = 0$. The behaviour for $\mu$ negative is obtained by observing that

$$\overline{\ln Z_L(-\mu)} = \overline{\ln Z_L(\mu)} + \alpha\mu L. \tag{2.14}$$

This relation simply follows by a change of variables $y = L - x$ in (2.4) and the statistical symmetry of the stochastic process $\xi$†. Equation (2.14) means that $\overline{\ln Z_L(\mu)} + \alpha\mu L/2$ is an even function of $\mu$ implying that by (2.5) all odd disorder averaged cumulants of order $m \geqslant 3$ vanish exactly. With equations (2.10)–(2.14) we have an exact result for $\overline{\ln Z_L(\mu)}$ for all $\mu$.

We are interested in the asymptotic behaviour $L \to \infty$ of the cumulants (2.5) for the Sinai case $\mu = 0$. Thus we need $\overline{\ln Z_L(\mu)}$ for large $L$ and small $\mu$. Due to the exponential in the integral (2.11) large $L$ implies that to leading order in $L$ only the immediate neighbourhood of $s = 0$ contributes. A complication arises because the term $T(\mu, s) = s\sinh(\pi s)/(\cosh(\pi s) - \cos(\pi\mu))$ in the integrand of (2.11) is discontinuous at the point of interest ($\mu = 0, s = 0$), since $\lim_{\mu\to 0}\lim_{s\to 0} T(\mu, s) = 0 \neq \lim_{s\to 0}\lim_{\mu\to 0} T(\mu, s) = 2/\pi$. Thus a naive saddle-point approximation would not serve our goal of obtaining the leading behaviour in $L$ of all coefficients in a small $\mu$ expansion of $\overline{\ln Z_L(\mu)}$. This goal is achieved by the decomposition

$$T(\mu, s) = \frac{2}{\pi}\frac{s^2}{s^2 + \mu^2} + R(\mu, s)$$

where the first, rational part captures the discontinuous behaviour of $T(\mu, s)$. The remainder $R(\mu, s)$ is analytic in $\mu$ and $s$, i.e. it can be expanded into a two-dimensional Taylor series around ($\mu = 0, s = 0$)

$$R(\mu, s) = \left(\frac{\pi}{6} + \frac{\pi^3}{120}\mu^2 + \mathrm{O}(\mu^4)\right)s^2 - \left(\frac{\pi^3}{360} + \frac{\pi^5}{1512}\mu^2 + \mathrm{O}(\mu^4)\right)s^4 + \cdots.$$

† A similar argument given in [29] yields $\partial\overline{\ln Z_L(\mu)}/\partial L = \overline{Z_L(-\mu)^{-1}}$ which in connection with our equation (2.14) leads to the interesting general relation $\partial\overline{\ln Z_L(\mu)}/\partial L = \overline{(Z_L(\mu))^{-1}} - \alpha\mu$.

Keeping only the rational part of $T(\mu, s)$ in (2.11) we can evaluate the corresponding integral exactly and expand the result into powers of $\mu$

$$\overline{\ln Z_L^{(c)}} \simeq -\frac{4}{\pi}\int_0^\infty \mathrm{d}s\,\frac{s^2}{(\mu^2+s^2)^2}\,\mathrm{e}^{-(\alpha L/4)(\mu^2+s^2)} \tag{2.15}$$

$$= -\frac{1}{\mu}\left[1-\mathrm{e}^{-(\alpha L/4)\mu^2}\sqrt{\frac{\alpha L}{\pi}\mu^2}+\frac{\alpha L}{2}\mu^2 - \left(1+\frac{\alpha L}{2}\mu^2\right)\mathrm{erf}\left(\sqrt{\frac{\alpha L}{4}\mu^2}\right)\right] \tag{2.16}$$

$$= -\frac{1}{\mu}+2\sqrt{\frac{\alpha L}{\pi}}-\frac{\alpha L}{2}\mu+\frac{1}{6\sqrt{\pi}}(\alpha L)^{3/2}\mu^2 - \frac{2}{15\sqrt{\pi}}\left(\frac{\alpha L}{4}\right)^{5/2}\mu^4+\mathrm{O}(\mu^6) \tag{2.17}$$

where $\mathrm{erf}(x) = (2/\sqrt{\pi})\int_0^x \mathrm{d}t\,\exp(-t^2)$ denotes the error function. The contributions to the integral (2.11) from the remainder $R(\mu, s)$ can also be evaluated exactly. It is easily checked that from terms $\sim s^{2k}$ in $R(\mu, s)$ one obtains contributions to the coefficients of $\mu^{2l}$, which increase at most as $L^{l+1/2-k}$, $k = 1, 2, \ldots$, while the leading terms as given in (2.17) increase as $L^{l+1/2}$ (see also (2.20)). This verifies explicitly that the remainder $R(\mu, s)$ contributes only lower-order corrections to the results (2.15)–(2.17).

The small $\mu$ expansion of the other part of $\overline{\ln Z_L(\mu)}$ given by (2.12) yields

$$\overline{\ln Z_L^{(0)}} = \frac{1}{\mu}-\ln\alpha+\gamma-\frac{\pi^2}{6}\mu+\mathrm{O}(\mu^2) \tag{2.18}$$

where $\gamma = 0.5772\ldots$ is the Euler constant. From (2.17) and (2.18) we can verify the cancellation of the singular terms $1/\mu$ and we can also infer $\overline{\ln Z_L(\mu=0)}$ of the Sinai model which is obtained as

$$\overline{\ln Z_L(\mu=0)} = 2\sqrt{\frac{\alpha L}{\pi}}-\ln\alpha+\gamma+\mathrm{O}(L^{-1/2}). \tag{2.19}$$

The same expression can be obtained directly with the aid of (2.9) and the expression for $\Phi_L(p)$ given by Oshanin *et al* [13] for $\mu = 0$. It was also found previously in [30]. Let us note that the terms of order $L^{-1/2}$ are the first corrections to the leading $L^{1/2}$ behaviour of the $\mu^0$ coefficient in (2.17) due to the remainder term $R(\mu, s)$. With $R(\mu = 0, s) \simeq \frac{1}{6}\pi s^2$ one obtains the explicit result $-\frac{1}{3}\pi^{3/2}(\alpha L)^{-1/2}$ for the correction term, which remarkably is also in full accordance with a replica calculation without symmetry breaking (see the appendix). In the following we will use the result (2.19) for the calculation of the Rényi entropies for this model.

By differentiating (2.17) with respect to $\mu$ the low-order cumulants are readily seen to be $\overline{K_L^{(1)}} = \overline{\langle x\rangle} = L/2$, $\overline{K_L^{(2)}} = \overline{\langle x^2\rangle - \langle x\rangle^2} \sim (1/3\sqrt{\alpha\pi})L^{3/2}$, etc. The somewhat surprising increase of the average variance $\overline{K_L^{(2)}}$ with increase of the system length $L$ is also confirmed numerically [26] and will be discussed later. The asymptotic large $L$ behaviour of the higher-order cumulants $\overline{K_L^{(m)}}$ is most easily derived by differentiating (2.15) twice with respect to $L$ resulting in a Gaussian integral, which can be evaluated and differentiated with respect to $\mu$ to any order. In this way one obtains

$$\overline{K_L^{(m)}} \sim (-1)^{(m/2)+1}\frac{1}{\pi}\frac{\Gamma((m-1)/2)}{m+1}\alpha^{(1-m)/2}L^{(1+m)/2} \tag{2.20}$$

for $m$ even, and $\overline{K_L^{(m)}} = 0$ for odd $m \geqslant 3$ in accordance with the symmetry (2.14). Note that $\alpha$ has the dimension of an inverse length, so that $\overline{K_L^{(m)}}$ has correctly the dimension $[\text{length}]^m$.

An alternative characterization of the equilibrium distribution is by the Rényi entropies $H_q$ of order $q$. In the discrete case they are defined as $H_q = (1-q)^{-1} \ln \sum_{i=1}^{L} \rho_i^q$. In the continuum limit one considers accordingly $H_q = (1-q)^{-1} \ln \int_0^L \mathrm{d}x\, \rho(x)^q$, which are actually relative Rényi entropies [23], which may, in contrast to the discrete case, also take negative values. The disorder averaged Rényi entropies $\overline{H_q}$ can be calculated from the relation

$$(1-q)\overline{H_q} = -q\overline{\ln Z_L(\alpha,\mu)} + \overline{\ln Z_L(\alpha q^2, \mu/q^2)}. \tag{2.21}$$

For the Sinai case $\mu = 0$ one obtains with (2.19) the exact result

$$\overline{H_q} = 2\frac{\ln q}{q-1} + \gamma - \ln\alpha + \mathrm{O}(L^{-1/2}) \tag{2.22}$$

valid for $q > 0$. For $q = 0$, one has the obvious result $H_0 = \ln L$ and for $q < 0$, $\overline{H_q}$ diverges as $L^{1/2}$. The exact result (2.22) confirms the $q$-dependence obtained in [22] by a mean-field-like approximation. Although for large $q$ one might expect differences between the discrete and continuum model, we find in [26] that the numerical data of [22] are very well described by our continuum result also with respect to the constant terms. This means that the conclusion in [22] that $\rho(x)$ is not multifractal because $\overline{H_q}/\ln L$ has only trivial limits for $L \to \infty$ is also established here in the continuum limit.

## 3. Summary and discussion

Random walks in random environments provide simple models for non-equilibrium processes in disordered environments. The connection with dynamical systems via Markov partitions [20, 21, 31] also makes them relevant for chaotic diffusion processes in the presence of quenched disorder.

We presented analytical results for the quenched average of the free energy and derived quantities such as averaged cumulants of the equilibrium distribution and Rényi entropies. One surprising result is that the variance of the equilibrium density scales with the length of the system as $L^{3/2}$. A simple argument for this behaviour is the following. Typically on any length scale $L$ the potential $V(x) \sim \int_0^x \xi(y)\mathrm{d}y$ has one absolute minimum and the equilibrium distribution is concentrated in the close neighbourhood of that minimum leading to a variance of order $\mathrm{O}(1)$. There is, however, a small probability that the system has a second relative minimum almost degenerate with the deepest one. Within a distance of order $\mathrm{O}(L)$ this occurs with a probability of order $\mathrm{O}(L^{-1/2})$ since the fluctuations of $V(x)$ increase with the system length as $L^{1/2}$. Such rare configurations contribute terms of the order $\mathrm{O}(L^2)$ to the variance of the equilibrium distribution. Together with its weight $\sim L^{-1/2}$ this leads to an averaged variance which scales as $L^{3/2}$. This means that atypical configurations dominate the quenched average $\overline{K_L^{(2)}}$. Similar arguments have been given by Parisi in the appendix of [25]. Note, however, that this simple argument cannot explain the scaling of the higher-order, even cumulants $\overline{K_L^{(m)}} \sim L^{(1+m)/2}$. To get access to the typical cumulants one would like to calculate quantities such as $\exp(\overline{\ln K_L^{(m)}})$. To do this analytically appears to be impossible. Numerical investigations of such quantities and also the numerical confirmation of the above results are presented in [26]. Finally, we would like to point out a connection to a closely related system. In the present work the particles were confined by hard walls at positions $x = 0$ and $x = L$. Alternatively, one can add to the random walk landscape a confining soft potential of the form $\lambda x^2$. The latter system was much studied, for example, in the context of random field Ising models [32–34]. For this system one finds the scaling $\overline{K_\lambda^{(2)}} \sim \lambda^{-1}$ and $\overline{\langle x^2\rangle} \sim \lambda^{-4/3}$. According to arguments by Parisi [25] $L$ and $\lambda$ are related by $L \sim \lambda^{-2/3}$ in accordance with $\overline{K_L^{(2)}} \sim L^{3/2}$ and the obvious result $\overline{\langle x^2\rangle} \sim L^2$ for the hard wall system. Applying the relation between $L$ and $\lambda$ and our result (2.20) to the higher-order cumulants of the soft potential system yields $\overline{K_\lambda^{(m)}} \sim \lambda^{-(m+1)/3}$ for even $m$.

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## Appendix

In this appendix it is briefly shown that the formula for the free energy of the Sinai model (2.19) also follows from a replica calculation without symmetry breaking. The starting point is the expression for $\overline{Z_L^n}$ for systems of length $L$ in the absence of an external field ($\mu = 0$) as given by Oshanin *et al* [13]. They find

$$\overline{Z_L^n} = \frac{2\alpha^{-n}}{\Gamma(n+\frac{1}{2})(\alpha L)^{1/2}} \int_0^\infty \mathrm{d}x \exp\left(-\frac{x^2}{\alpha L}\right) \sinh^{2n}(x). \tag{A.1}$$

The free energy $\overline{\ln Z_L}$ is obtained by the replica trick from $\overline{\ln Z_L} = \lim_{n\to 0}(1/n)(\overline{Z_L^n} - 1)$, or equivalently as

$$\begin{aligned}
\overline{\ln Z_L} &= \frac{\partial}{\partial n}\overline{Z_L^n}\bigg|_{n=0} \qquad\qquad (A.2)\\
&= -\ln\alpha - \Psi\left(\frac{1}{2}\right) + \frac{4}{\sqrt{\pi}}\int_0^\infty \mathrm{d}y \exp(-y^2)\ln\sinh(y\sqrt{\alpha L})\\
&= -\ln\alpha + \gamma + 2\sqrt{\frac{\alpha L}{\pi}} + \frac{4}{\sqrt{\pi}}\lim_{R\to\infty}\int_0^R \mathrm{d}y\exp(-y^2)\ln(1-\mathrm{e}^{-2y\sqrt{\alpha L}})\\
&= -\ln\alpha + \gamma + 2\sqrt{\frac{\alpha L}{\pi}} + \frac{2}{\sqrt{\alpha L\pi}}\lim_{R\to\infty}\int_{\exp(-2R\sqrt{\alpha L})}^1 \mathrm{d}x\\
&\quad\times\exp\left(-\frac{\ln^2 x}{4\alpha L}\right)\frac{1}{x}\ln(1-x).
\end{aligned}$$

The last integral becomes, for large $L$, asymptotically equal to $-\pi^2/6$ showing that the replica symmetric result is identical to that of equation (2.19). This result was also found independently by Comtet *et al* [29]. They prove, in addition, the non-trivial fact that (A.2) is not only an asymptotic but an exact expression.

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